

# Reduced Basis-Methods on Parametrized Geometries

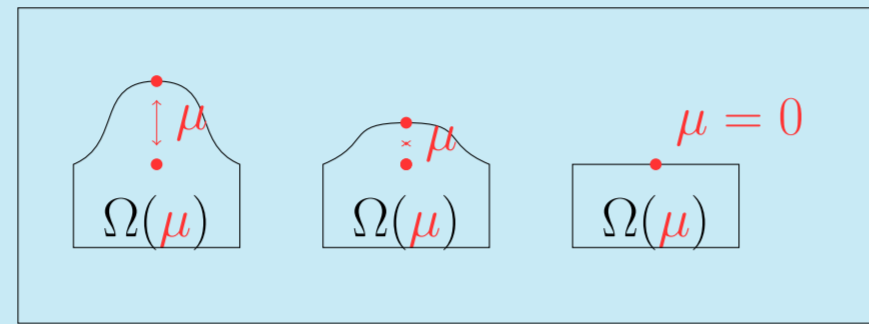
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## Abstract

We want to discuss **parametrized partial differential equations** (P<sup>2</sup>DEs) for parameters that describe the geometry of the underlying problem. One can think of applications in control theory and shape optimization which depend on time-consuming parameter-studies of such problems. Therefore, we want to reduce the order of complexity of the numerical simulations for such P<sup>2</sup>DEs.

**Reduced Basis (RB) methods** are a means to achieve this goal. These methods have gained popularity over the last few years for model reduction of finite element approximations of elliptic and instationary parabolic equations. We present a RB method for parabolic problems with general geometry parameterization and finite volume (FV) approximations. Experimental results are presented for a simple test problem.



## Test problem and geometry transformation

We focus on a two dimensional instationary heat equation as a model problem:

**Problem 1** (Instationary heat equation). For every  $\mu \in \mathcal{P}$  we want to determine a solution  $u(x, t; \mu)$  on a polygonal domain  $\Omega(\mu) \subset \mathbb{R}^2$  for all times  $t \in \mathbf{T} := [0, T_{\max}]$ ,  $T_{\max} > 0$ , which satisfies the equations

$$\begin{aligned} \partial_t u(x, t; \mu) - a(\mu) \Delta u(x, t; \mu) &= 0 & \text{in } \Omega(\mu) \times \mathbf{T} & \quad (1a) \\ u(x, 0; \mu) &= u_0(x; \mu) & \text{in } \Omega(\mu). & \quad (1b) \end{aligned}$$

and certain boundary conditions.

In order to apply the RB method, however, the function space must not depend on the parameter. Therefore, we reformulate the problem on a reference domain, which results in a convection-diffusion-reaction equation with an (in general) anisotropic diffusion tensor.

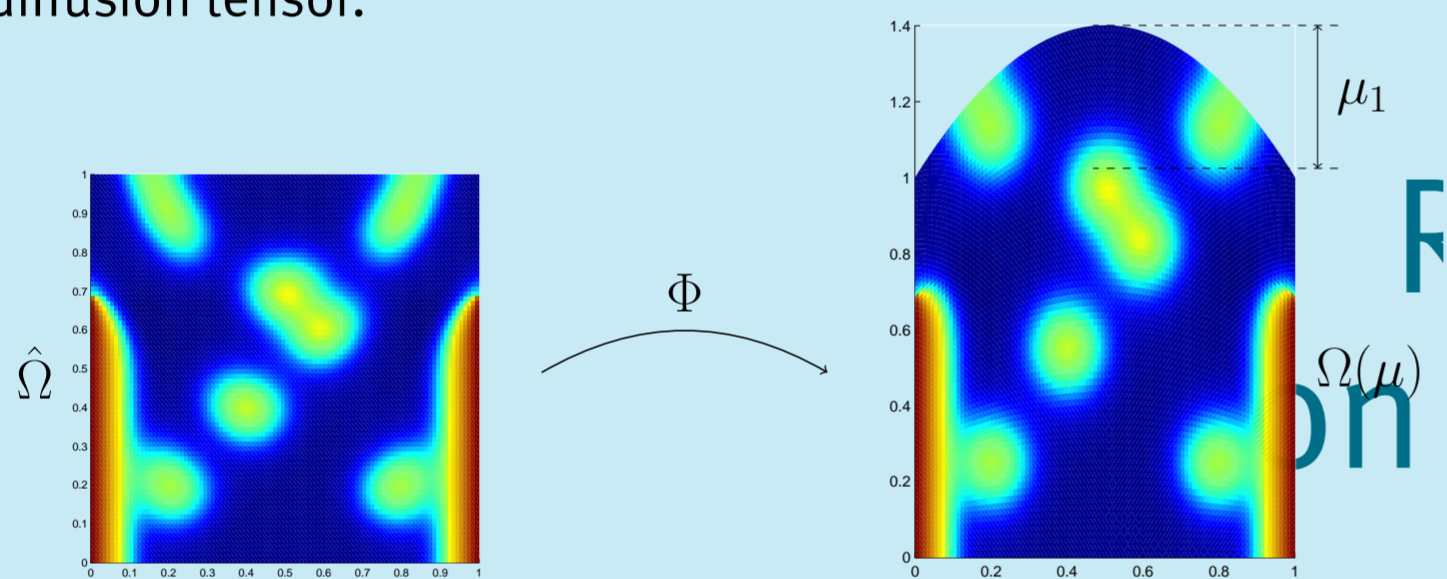


Figure: Illustration of a geometry transformation

We select an arbitrary parameter  $\hat{\mu} \in \mathcal{P}$  that defines the reference domain  $\hat{\Omega} := \Omega(\hat{\mu})$ . It is assumed that for every parameter  $\mu$  there exists a diffeomorphism  $\Phi(\mu) : \hat{\Omega} \rightarrow \Omega(\mu)$ . By transforming the heat equation onto the reference domain, we get the following

**Lemma 2** (Geometry transformation). Let  $u$  be a solution of problem 1. Then the function  $\hat{u}(\hat{x}, t) := u(\Phi(\hat{x}), t; \mu)$ , with coordinates  $\hat{x} := \Phi^{-1}(x)$  on the reference domain, is a solution of the equivalent convection-diffusion-reaction equation

$$\partial_t \hat{u} - a(\mu) \nabla_{\hat{x}} \cdot (G G^t \nabla_{\hat{x}} \hat{u}) + a(\mu) \nabla_{\hat{x}} \cdot (v \hat{u}) - a(\mu) (\nabla_{\hat{x}} \cdot v) \hat{u} = 0 \quad \text{in } \hat{\Omega} \times \mathbf{T}. \quad (2)$$

with notations

$$\tilde{v}(\hat{x}) := \begin{pmatrix} \partial_{\hat{x}_1} G_{11}(\hat{x}) & \partial_{\hat{x}_1} G_{12}(\hat{x}) \\ \partial_{\hat{x}_2} G_{21}(\hat{x}) & \partial_{\hat{x}_2} G_{22}(\hat{x}) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v(\hat{x}) := G(\hat{x}) \tilde{v}(\hat{x}), \quad (3)$$

with  $G(\hat{x}) = (G_{ij}(\hat{x}))_{i,j=1,2}$  being the Jacobi matrix of the inverse geometry transformation

## References

- A. Patera and G. Rozza. *Reduced Basis Approximation and A Posteriori Error Estimation for Parametrized Partial Differential Equations*. Version 1.0, Copyright MIT 2006, to appear in (tentative rubric) MIT Pappalardo Graduate Monographs in Mechanical Engineering
- O. Drblíková and K. Mikula, *Convergence Analysis of Finite Volume Scheme for Nonlinear Tensor Anisotropic Diffusion in Image Processing*. SIAM J. Numer. Anal., 46, 37–60, 2007
- B. Haasdonk und M. Ohlberger *Reduced Basis Method for Finite Volume Approximations of Parametrized Linear Evolution Equations* M2AN, Math. Model. Numer. Anal. 42, 277–302, 2008
- B. Haasdonk, M. Ohlberger und G. Rozza. *A Reduced Basis Method for Evolution Schemes with Parameter-Dependent Explicit Operators* ETNA, Electronic Transactions on Numerical Analysis, vol. 32, 2008
- M. Barrault, M., Y. Maday, N. Nguyen and A. Patera, *An 'empirical interpolation' method: application to efficient reduced-basis discretization of partial differential equations* C. R. Math. Acad. Sci. Paris Series I, 2004, 339, 667-672
- M. Drohmann, B. Haasdonk and M. Ohlberger, *Reduced Basis Method for Finite Volume Approximation of Evolution Equations on Parametrized Geometries* Proceedings of ALGORITHMY 2009, 111–120

## Model reduction with Reduced Basis-Methods

### Scenario:

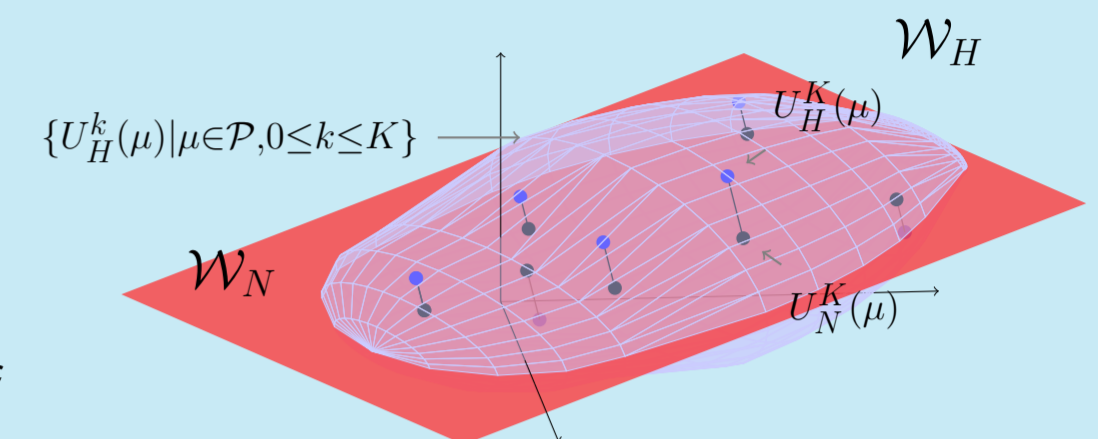
- Parametrized partial differential equations: shape, material or control parameters  $\mu \in \mathcal{P} \subset \mathbb{R}^p$

$$\partial_t u(\mu) + \mathcal{L}(\mu)[u(\mu)] = 0 \quad + \text{initial and boundary conditions}$$

- Simulation requests need to be answered **rapidly** or **repeatedly** for many different parameters, e.g. design optimization, control, parameter estimation, real-time applications.

### Goals:

- Automatic computation of reduced basis for approximation of numerical simulations  $U_H(\mu)$  by reduced simulation  $U_N(\mu)$
- Offline-Online decomposition of computations
- rigorous a-posteriori error estimators

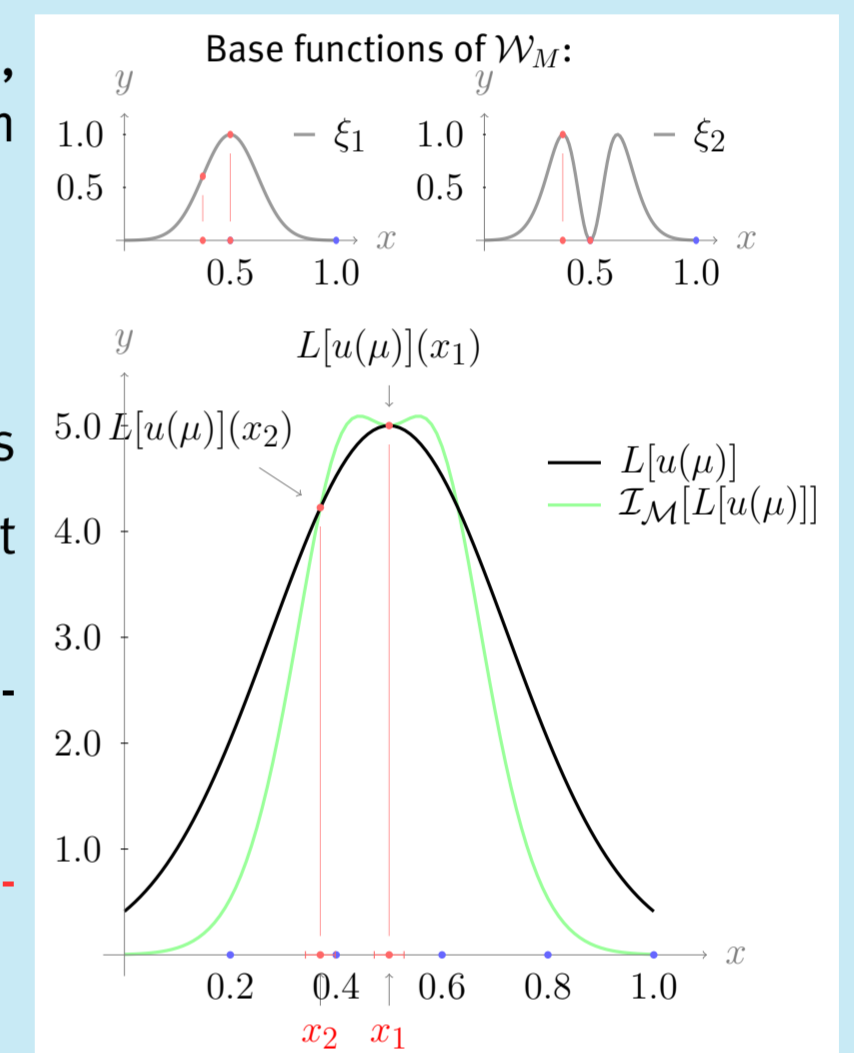


## Empirical interpolation

- Reduced simulation in RB space is possible, if the discrete operators and the problem data functions
  - are linear and
  - depend affinely on the parameter.
- If not  $\Rightarrow$  **Empirical interpolation** of operators

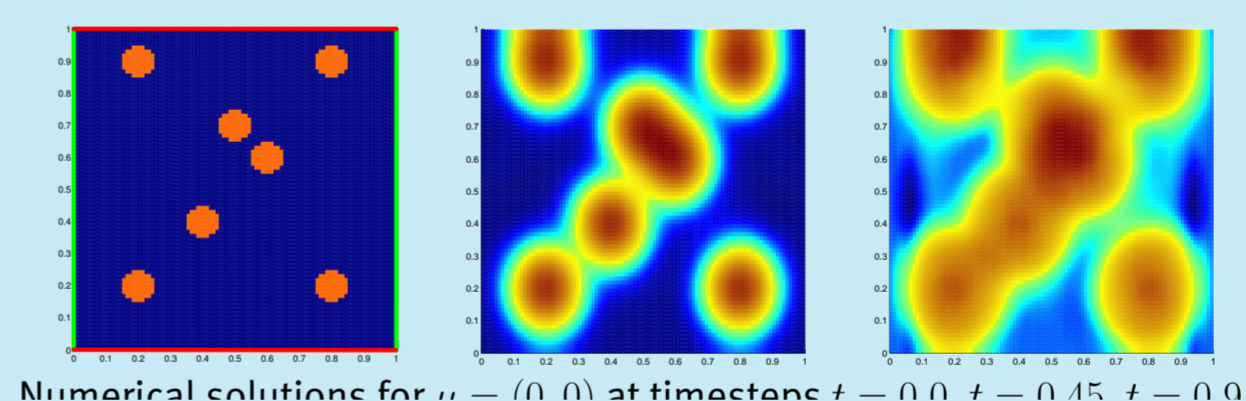
**Idea:** Approximate operator with few point evaluations

- Build collateral RB space of operator evaluations  $\mathcal{W}_M := \text{span} \left\{ L(\mu_i) [U_H^{k_i}(u_i)] \right\}_{i=1}^M$
- Interpolate efficiently if operator is **localised**, by  $\mathcal{I}_M[L(\mu)[U]] := \sum_{m=1}^M L(\mu)[U](x_m) \xi_m(x)$

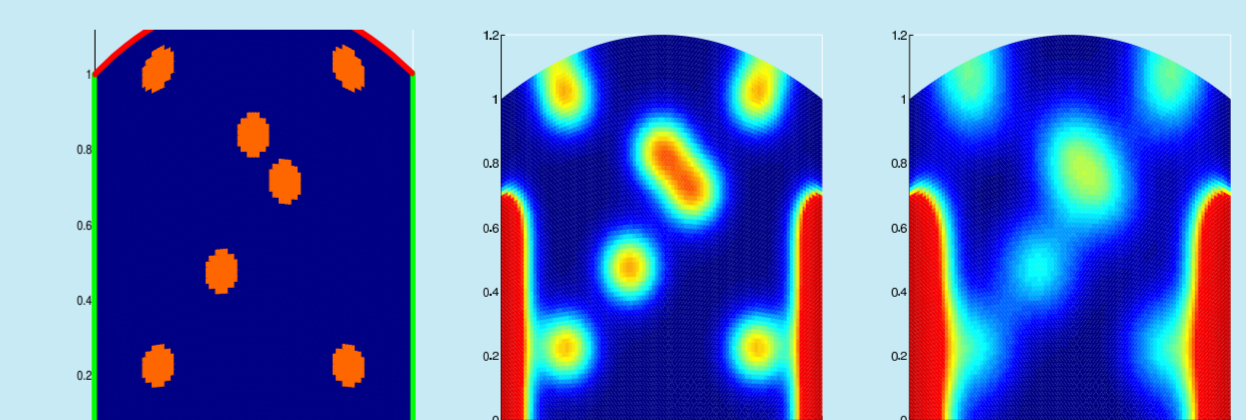


## Implementation and results

The implementation of the experiments is integrated into our package RBmatlab that provides FV discretizations, algorithms for RB generations, empirical interpolation and a demonstration GUI for online simulations. We chose a high dimensional function space with 8000 degrees of freedom and used 16 detailed simulations during the offline phase.



Time gain factor:  
 $\approx 5$



Average approximation error:  
 $\approx 10^{-3}$

Numerical solutions for  $\mu = (0.2, 0.2)$  at time steps  $t = 0.0, t = 0.45, t = 0.9$ .

[http://www.uni-muenster.de/math/num/ag\\_ohlberger](http://www.uni-muenster.de/math/num/ag_ohlberger)

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